Midsemestral

Elementary Number Theory

Instructor: Ramdin Mawia Marks: **30** **Time:** September 22, 2023; 14:00–17:00.

Attempt SEVEN problems. Each question carries 5 marks. The maximum you can score is 30.

Induction

1. Prove the following formula by using mathematical induction: **5**

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}, \text{ for all } n \geqslant 1.$$

Let $t_n = 1 + 2 + \cdots + n$ denote the n-th triangular number. Using the above formula, compute the values n for which t_n divides the sum $t_1 + t_2 + \cdots + t_n$.

Primes, Divisibility and Congruences

2. State whether the following statements are true or false, with complete justifications: **5**

   i. An integer $n > 4$ is composite if and only if n divides $(n-1)!$

   ii. An odd integer $n > 1$ is composite if and only if it can be expressed as a sum of three or more consecutive positive integers.

3. If $\gcd(a, b) = 1$, determine all possible values of $\gcd(a + 6b, 6a + b)$. Give examples where these values are attained. **5**

4. Find the last three digits of 23^{2023} in the usual decimal notation. Justify all the steps. **5**

5. Let p be a prime. For any positive integer n, show that the largest positive integer e such that $p^e \mid n!$ is given by **5**

$$e = \sum_{j=1}^{\infty} \left[\frac{n}{p^j} \right].$$

Using this formula or otherwise, prove that if $n > 2$, then there exists a prime p satisfying $p \mid \binom{n}{k}$ for all $1 \leqslant k \leqslant n-1$ if and only if $n = p^r$ for some positive integer r.

Primitive roots and Quadratic residues

6. State the Quadratic Reciprocity Law. Use it to evaluate the Legendre symbol $(123456/951943)$. Note that 951943 is a prime. You may use your pocket calculator. **5**

7. Find integers a and b such that $5a^2 + 11^3 b = 37$. [1] **5**

8. Given any prime p, show that, for some choice of $n > 0$, p divides the integer **5**

$$(n^2 - 2)(n^2 - 3)(n^2 - 6).$$

Can we say the same thing of $(n^2 - 2)(n^2 - 3)(n^2 - 5)$? Justify.

9. Does there exist a rectangle with integral sides satisfying the following two conditions? a) Its length is 14 units longer than its breadth; b) Its area is congruent modulo 2023 to its perimeter. If there exists such a rectangle, find the one with shortest perimeter. You may use your pocket calculator. **5**

Arithmetic functions

10. Let $\tau(n)$ denote, as usual, the number of positive divisors of $n \in \mathbb{Z}^+$. Prove the following: **5**

   i. $\prod_{d \mid n} d = n^{\tau(n)/2}$.

   ii. $\tau(m)\tau(n) = \sum_{d \mid (m,n)} \tau(mn/d^2)$ where the sum runs through the positive divisors d of (m, n), the gcd of m and n.

[1] Enough to find one value of a and one value of b.